DETERMINING THE OPTIMAL POINT OF SAIL TO REACH AN UPWIND MARK IN A DINGHY

Introduction:

Dinghy sailing is a water sport in which the wind is used to propel a small boat. Downwind sailing makes sense, air molecules hitting the sail exert a net force upon it accelerating the boat up to the true wind speed at which point the sail will exert a braking force as collisions occur on the other side of it. However, since the early days of marine navigation, boat designs have made it possible to sail upwind as well.

As part of the school sailing team, I have experienced upwind sailing and I know that the point of sail (the angle to the wind) can affect the boat speed. Additionally, further research, reading Frank Bethwaite's, "High Performance Sailing", has elucidated the physics behind how modern dinghies work.

The sail of a dinghy is shaped like and acts as an aerofoil generating lift to propel the boat. The lift generated is always perpendicular to the apparent wind (the velocity of air molecules meeting the aerofoil) but the effect of the hydrofoils on the boat (daggerboard & rudder) which reduce sideways drift, mean that only the component of the lift force in the direction of travel of the boat needs to be considered.

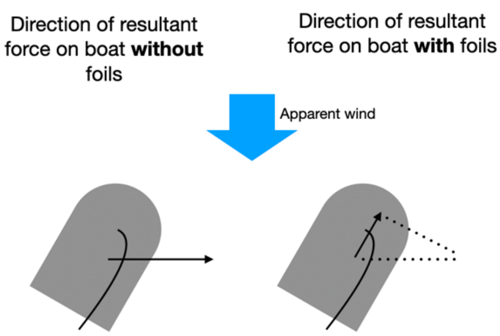

Figure 1

Objectives

This investigation will produce a mathematical model to determine the optimal point of sail to Reach an Upwind Mark in a Dinghy. It will also look at how this optimal point of sail depends on the windspeed and factors affecting drag or lift.

Assumptions

To simplify the task of working out the effect of point of sail on upwind boat speed, some assumptions have been made to eliminate variables. It is assumed that the boat will not drift sideways by a noticeable amount and so this will be altogether ignored. The drag coefficient will remain constant in each calculation whereas many boats will have a decreasing drag coefficient as they can plane upwind (planing is where the angle of attack of the boats hull creates lift raising the boat out of the water and decreasing the area upon which drag can act). It will also be assumed that the boat has been kept flat by the crew and that the sail is perfectly trimmed.

Hypothesis

Based on my personal experience sailing a variety of different boats, the optimal upwind point of sail is about 30-40 degrees from the wind. I think the lift constant/drag constant ratio for a particular boat will have an appreciable effect since different types of boat have varying abilities to sail upwind. I expect the boat velocity will not be directly proportional to true windspeed and so the true windspeed is expected to have some effect on the optimum angle.

Maths behind wind

Wind velocity is relative. An observer stationary relative to the earth experiences the true wind velocity, a boat moving at any velocity on the water will experience the apparent wind, the true wind velocity minus the boat velocity. In this way, a boat moving upwind will experience a higher apparent wind than a boat moving downwind in the same true wind. It also explains how boats can sail faster than the wind: they only sail faster than the true wind, the apparent wind may be much higher.

Lift

Lift is an important physical force and is proportional to the square of the speed. The shape of the sail forces an area of lower pressure to form over the top surface of the sail and the pressure gradient between the top and bottom of the aerofoil creates an upwards force called lift perpendicular to the flow of air molecules over the sail. In contrast to an aeroplanes wing

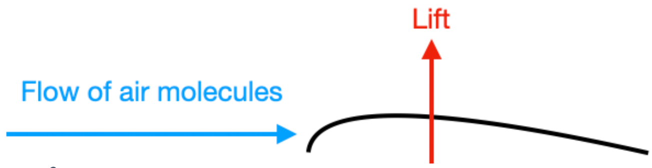

Figure 2

(which also produces lift in this way), in the case of a dinghy sail (when the dinghy is level), the lift is parallel to the water's surface.

Constructing a formula

In this section, the different forces & velocities on a boat will be considered, resolved, and then built into an equation to find the terminal velocity of the boat at a given angle in given wind conditions. The approach I take to this problem will use a computer to approximate the values and see whether the trend is largely impacted by the ratio between drag and lift coefficients.

In this model, the boat will start at rest but will be pointing in a fixed angle, θ which will be represented by the vector $\tilde{B}_b$ of magnitude 1.

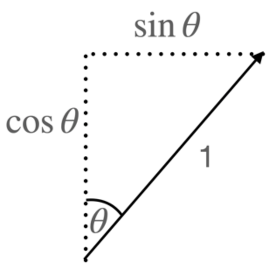

Figure 3

$$\tilde{B}_b = \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}$$

The true wind will have a constant velocity $\tilde{V}_T$, in this example it will be 18 knots. For simplicity it will always act along the y axis in a negative direction, it was modelled this way as it eliminated the direction of the wind as a factor in calculations.

$$\tilde{V}_T = \begin{bmatrix} 0 \\ -18 \\ 0 \end{bmatrix}$$

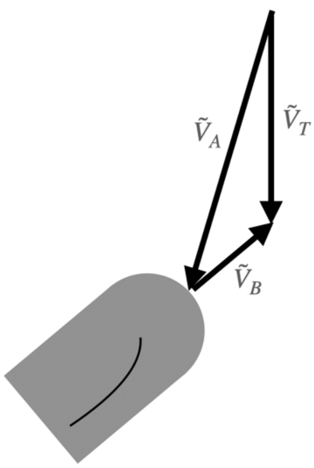

Figure 4

The driving force for the boat must be considered: the apparent wind. The true wind, $\tilde{V}_T$, the wind which a stationary observer would experience is used as the frame of reference and is shown coming vertically downwards. This velocity will be independent so that different wind conditions can be evaluated. However, a sailing boat is rarely stationary and as such the boat's velocity, $\tilde{V}_B$, must be accounted for in the velocity of the apparent wind, $\tilde{V}_A$ (the direction and magnitude of the air running over the sails shown in Figure 4).

$$\tilde{V}_A = \tilde{V}_T - \tilde{V}_B$$

Next, the lift is calculated from the equation below. Since the values of ρ (air density), A (sail area) and C_l (lift coefficient) are constants and C_l is unknown (but could be experimentally determined), the equation can be simplified with a single constant, K_l to reflect this.

$$L = \frac{\rho \left|\tilde{V}_A\right|^2 A C_l}{2}$$

$$L = K_l \left|\tilde{V}_A\right|^2$$

However, this yields a scalar product for L. To get the direction of the lift, we use the knowledge that a sail produces lift at a right angle to the direction of airflow. With this knowledge, we can create a new vector in this direction by the cross product of $\tilde{V}_A$ and $\tilde{U}_\uparrow$ (a unit vector pointing straight up out of the plane) which produces a vector at a right angle to $\tilde{V}_A$ with the same magnitude. Dividing this vector by $|\tilde{V}_A|$ creates a unit vector in the direction of the lift. Multiplying this new unit vector by L (the magnitude of the lift) gives the vector for lift, $\tilde{\xi}$.

$$\tilde{U}_\uparrow = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{\xi} = \frac{(\tilde{V}_A \times \tilde{U}_\uparrow)L}{|\tilde{V}_A|}$$

If the lift alone determined the course of the boat, it would be impossible to go upwind at all since the boat would drift sideways in the direction of the lift provided by the sail. However, the boat's foils: the daggerboard and rudder, prevent the boat from drifting and mean only $\tilde{T}$ (thrust), the component of $\tilde{\xi}$ which is colinear with $\tilde{B}_b$ is considered. This is calculated as:

$$\tilde{T} = \tilde{B}_b|\tilde{\xi}|\cos(\phi)$$

where ϕ is the angle between the lift and the boat bearing.

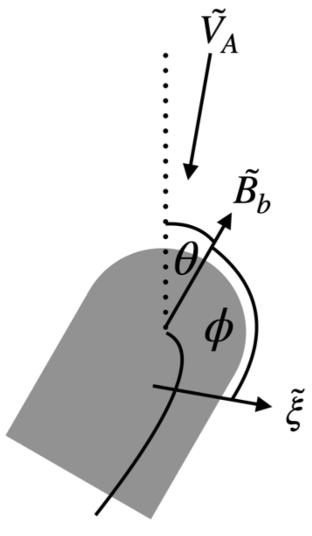

Figure 5

Knowing both the value of $\tilde{B}_b$ and $\tilde{\xi}$, the angle can be calculated as:

$$\phi = cos^{-1}(\frac{\tilde{\xi} \cdot \tilde{B}_b}{|\tilde{\xi}||\tilde{B}_b|})$$

Hence the force $\tilde{T}$ can be expressed by multiplying the unit vector $\tilde{B}_b$ by the calculated magnitude of the force:

$$\tilde{T} = \tilde{B}_b |\tilde{\xi}| \frac{\tilde{\xi} \cdot \tilde{B}_b}{|\tilde{\xi}||\tilde{B}_b|}$$

The only other force which will be considered is hydrodynamic drag, although there will also be aerodynamic drag it will be small in comparison to the effect of the water since water is denser. The equation for this force is shown and knowing that it opposes the movement of the boat its direction can be calculated using $\tilde{B}_b$. The magnitude can be determined in a similar manner to the lift using a drag constant K_D to replace the constants ρ_w (water density), A_h (hull area) and C_d (drag coefficient).

$$D = \frac{\rho_w |\tilde{V}_B|^2 A_h C_d}{2}$$

$$D = K_D |\tilde{V}_B|^2$$

$$\tilde{D} = -\tilde{B}_b D$$

Now, the net force acting on the boat is found as the vector sum of the individual forces.

$$\tilde{T} + \tilde{D} = \tilde{F}_n$$

Force is equal to mass times acceleration (mass will be set to 1 unless otherwise stated), hence, to find the acceleration of the boat at any point in time.

$$\frac{\tilde{F}_n}{m} = \tilde{a}$$

At which point the new boat velocity can be defined as a recurrence relationship which can be iterated by a computer program to approximate the results:

$$\tilde{V}_{Bn} = \tilde{V}_{Bn-1} + \tilde{a}t$$

By substituting in previously defined variables, the equation can be expanded to show the values which must be defined to calculate the recurrence relation.

$$\tilde{V}_{Bn} = \tilde{V}_{Bn-1} + \frac{t}{m}\left(\begin{bmatrix}\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix}\left|\left|\tilde{V}_T - \tilde{V}_{Bn-1}\right|^2 K_l \frac{\tilde{U}_\uparrow \times (\tilde{V}_T - \tilde{V}_B)}{\left|\tilde{V}_T - \tilde{V}_B\right|}\right|\left(\frac{\left|\tilde{V}_T - \tilde{V}_B\right|^2 K_l \frac{\tilde{U}_\uparrow \times (\tilde{V}_T - \tilde{V}_B)}{\left|\tilde{V}_T - \tilde{V}_B\right|} \cdot \begin{bmatrix}\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix}}{\left|\left|\tilde{V}_T - \tilde{V}_B\right|^2 K_l \frac{\tilde{U}_\uparrow \times (\tilde{V}_T - \tilde{V}_B)}{\left|\tilde{V}_T - \tilde{V}_B\right|}\right|\left|\begin{matrix}\sin(\theta)\\ \cos(\theta)\\ 0\end{matrix}\right|}\right) - \begin{bmatrix}\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix}\left|\tilde{V}_{Bn-1}\right|^2 K_D\right)$$

Finding the algebraic infinite sum of this recurrence relation is beyond the scope of this investigation and so instead, the 300th term will be calculated programmatically (a point at which most sequences were found to plateau.

Were it possible to find the infinite sum of this sequence, which is expected to converge for the range $\theta \in \{\mathbb{Z}, 0 \leq \theta \leq 90\}$, the max speed must be found for a given angle. The upwind component of the windspeed can be calculated as a function of θ.

$$u(\theta) = \left|\tilde{V}_{B\infty}(\theta)\right| \cos(\theta)$$

With this function defined, the maximum point can be found from the point of the derivative where the gradient is 0. This can be used to find the angle θ at which the upwind boat speed is greatest.

$$\frac{du(\theta)}{d\theta} = 0$$

Numerical analysis

A computer program (contained in the index) was written, then employed to conduct an analysis based on numerical approximations to determine whether the functions modelled match what is observed. The program was used to generate the following graphs. (Figures 6, 8-13)

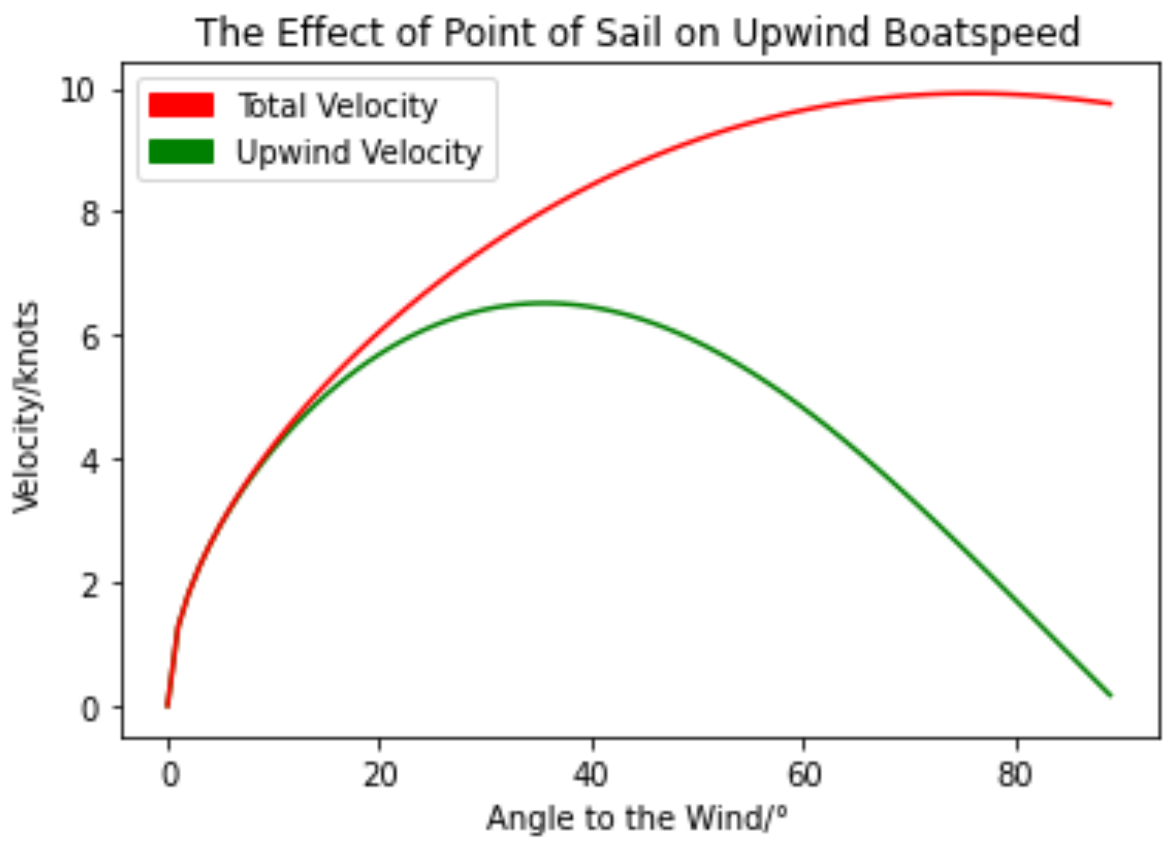

Figure 6

Figure 7: Input values: $K_L = 0.01,\ K_D = 0.1,\ |\tilde{V}_T| = 30,$ Optimal angle generated: $\theta = 35°$.

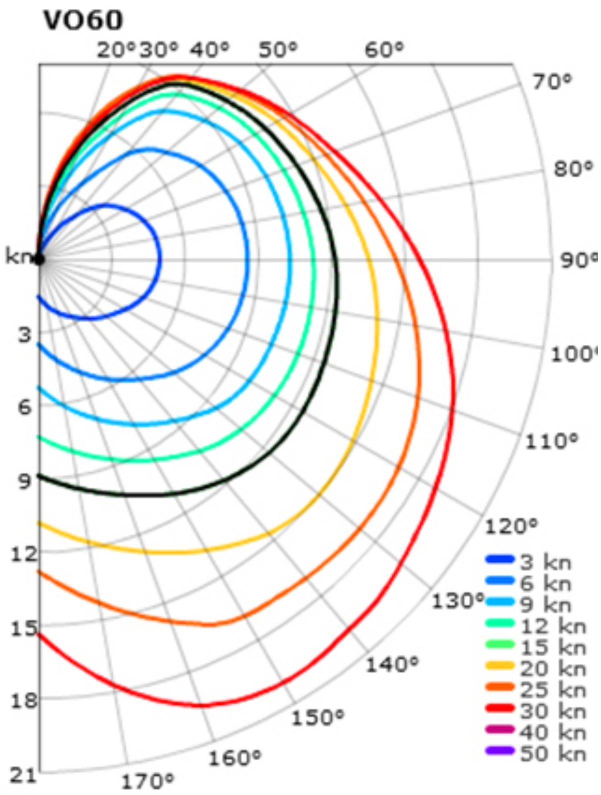

Figure 7

In a test of the code (Figure 6), the relationship was as expected with the boat velocity being small at low angles to the wind and increasing up to a point around 30-40°. In figure 6, a representative graph is shown for a lift constant of 0.01 and a drag constant of 0.1 in a true wind of 30 knots. In this initial simulation the maximum upwind velocity was 6.5 knots at 36° to the wind. This matches what I expected based on my experience sailing and real world data from the Volvo Ocean Racer 60 boats in figure 7 (Sailboat speed versus sailing angle - Sailing Blog by NauticEd, 2022) Where the max upwind velocity was achieved between 35-40°.

Since the lift constant and drag constant cannot be determined without experiments, exact values cannot be used. The extent to which these values impact the angle can be determined by studying their behaviour in a variety of conditions. Based on our knowledge of the drag constant is likely to be larger than the lift constant. This is because these constants are defined as a function of the density of their respective mediums. Air is 1/1000 as dense as water and this will make the largest contribution to the difference between the two constants.

$$K_D = \frac{\rho_w A_h C_d}{2}$$

$$K_l = \frac{\rho_a A_s C_l}{2}$$

Using representative values, It was found that the lift and drag constants did not change the results very much. Where $K_L \leq K_D$ and $1 \geq K_L, K_D > 0$, maximum upwind velocity was found between 35-45° (shown in Figures 8 & 9). Values outside this range caused parts of the simulation to behave erratically as the program struggled to handle with extremely large or small numbers.

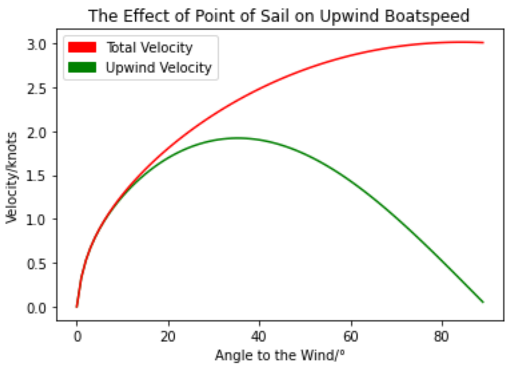

Figure 6

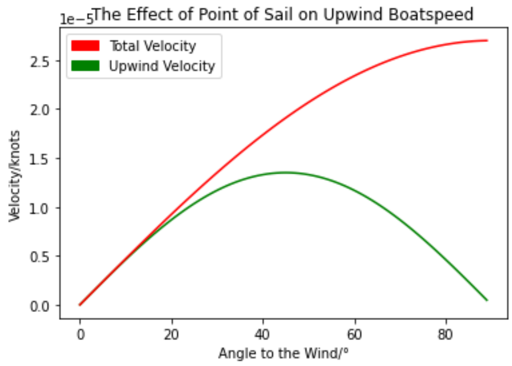

Figure 7

Figure 8: Input values: $K_L = 0.0001$, $K_D = 0.01$, $|\tilde{V}_T| = 30$, Optimal angle generated: $\theta = 35°$.

Figure 9: Input values: $K_L = 0.000001$, $K_D = 0.000001$, $|\tilde{V}_T| = 30$, Optimal angle generated: $\theta = 45°$.

These results seemed reasonable so to test the extreme behaviours of the simulation the boat was given no drag by setting the drag constant to zero. We expect this to result in a much faster boat possibly accelerating indefinitely and since it will sail principally under its own apparent wind the angle to the true wind at which it most efficiently sails upwind is also expected to change becoming more effective at higher angles from the wind. The simulation of this scenario in figure 10 reflects these predictions although the total velocity decreases at angles close to 90° since not enough iterations have been performed for the boat to come to equilibrium velocity.

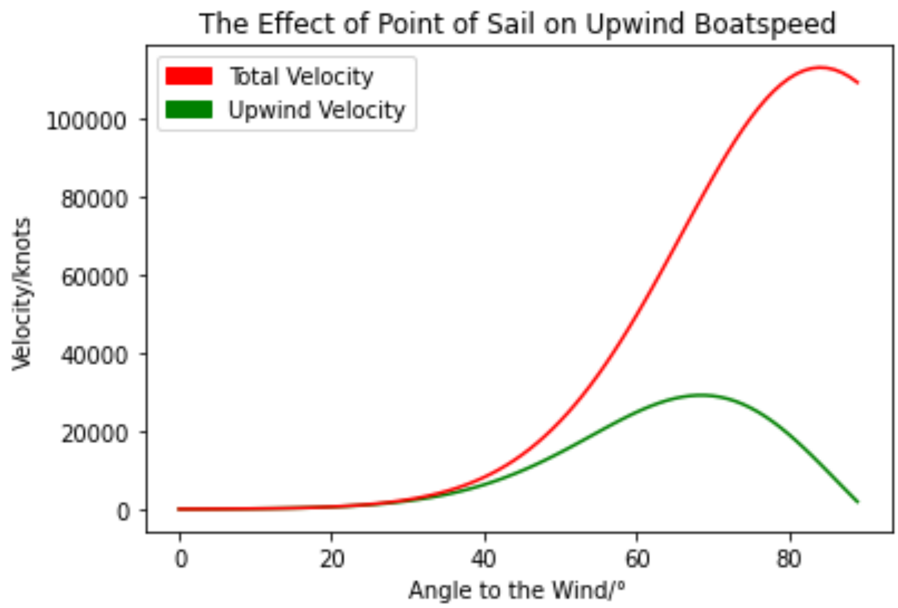

Figure 8

Figure 10: Input values: $K_L = 0.001$, $K_D = 0$, $|\tilde{V}_T| = 30$, Optimal angle generated: $\theta = 69°$.

Figure 11 shows the effect of setting the lift constant to zero: the velocity of the boat remains at zero for the duration of the simulation since with no lift there is no force to accelerate the boat. In reality, the boat would slowly start to drift backwards as the much smaller aerodynamic drag forces built up on the surfaces of the boat but since this force will be so small it does not need to be considered in practical applications of this simulation.

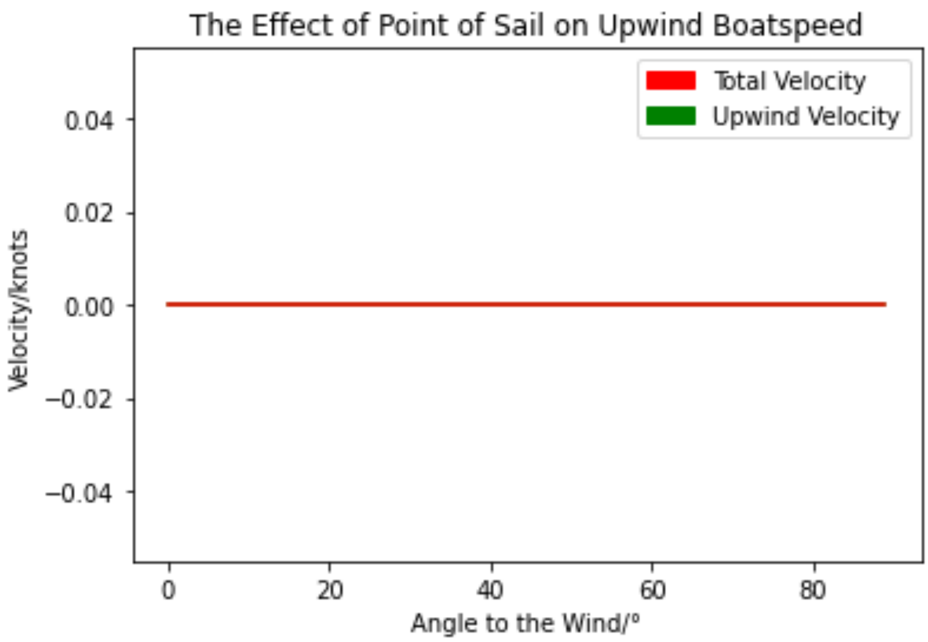

Figure 9

Figure 11: Input values: $K_L = 0$, $K_D = 0.01$, $\left|\tilde{V}_T\right| = 30$, Optimal angle generated: $\theta =$ no difference with variation of θ

Another variable which has a large potential effect on the optimal point of sail is the windspeed. Using modified code, Figure 12 shows that as the windspeed increases it becomes more effective to sail at a smaller angle to the true wind. Once again, the points of sail remained in the range 35-45°. The plot is non-differentiable as the program only approximated the optimal angle to the wind to the closest 0.5° but it is still effective in showing the overall trend for a particular boat across a range of conditions.

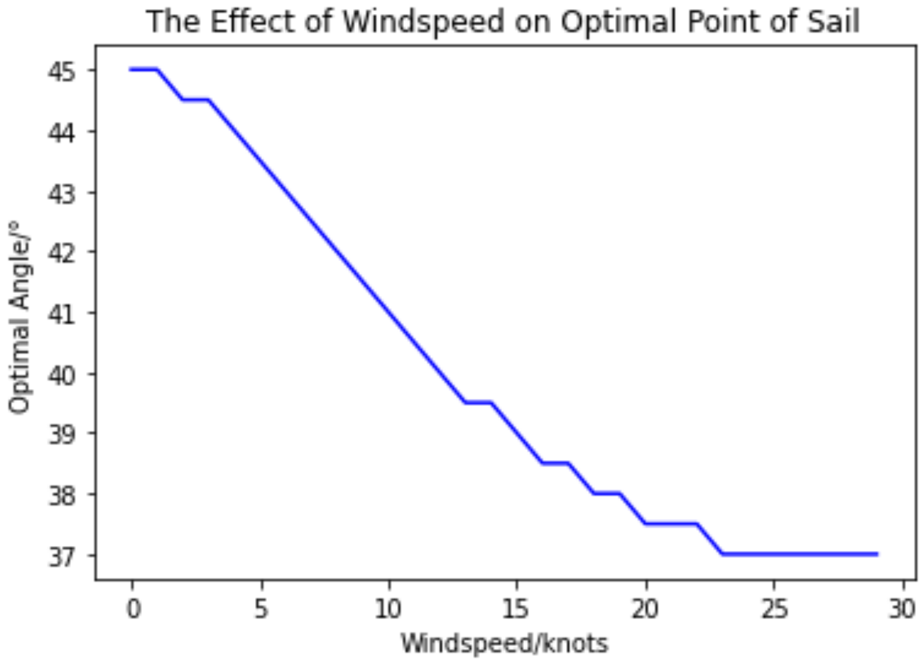

Figure 10

Figure 12: Input values: $K_L = 0.0005$, $K_D = 0.001$

Finally, the mass of the boat can be considered. The plot in figure 13 shows that a lighter boat can sail closer to the true wind given the same lift and drag constants. However, this does not fully reflect the advantages of having a higher boat mass such as the addition of a keel which can help keep the boat flat in high wind velocities and the fact that heavier boats will usually be larger with bigger sails (larger lift constant). The optimal point of sail is limited to the range 35-45°.

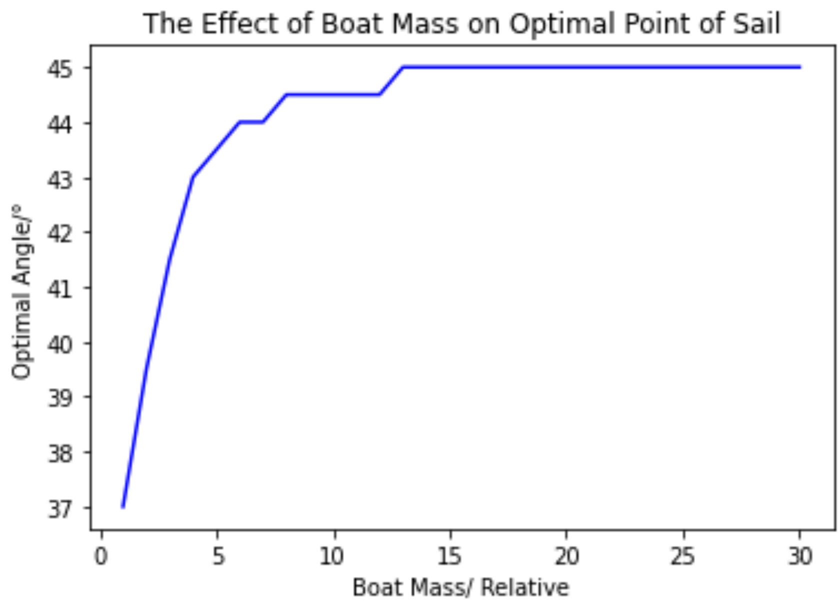

Figure 11

Figure 13: Input values: $K_L = 0.0005$, $K_D = 0.001$, $|V_T| = 30$

Under simulated conditions, the optimal sail angle of a dinghy will be restricted between 35° and 45° a result which matches the hypothesis as well as observational data. Boat designs and sailing conditions are variable and by measuring accurate hydrodynamic and aerodynamic data it would be possible to further optimise this simulation to make predictions about specific boats as well as predicting general trends.

Evaluation

Although the simulation has yet to produce results revolutionising everyday sailing, it does highlight some key trends which boat builders are already considering in their designs. The importance of increasing lift while reducing drag and boat mass have been known for centuries and the recent popularity of lightweight foiling boats including the International Moth (Figure 14) which has extremely low drag when it is lifted out of the water by specialised underwater wings shows how modern technology has allowed these principles to be taken to the extremes.

Figure 12

While this investigation looked at boats as two-dimensional with perfect balance, boats have a heeling moment when going upwind (Figure 15) which makes the sails less effective at converting wind to lift as it reduces their profile. Taking this into account would increase the optimal angle to the wind since the amount of heel increases as you sail closer to the wind. Alternatively, assuming the crew have a high enough mass to keep the boat flat, there would be little effect on the current simulation.

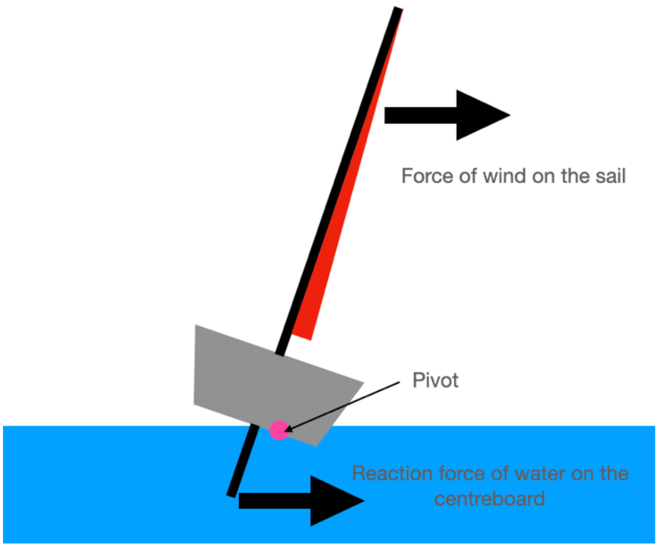

Figure 13

The simulation assumes that the boat does not drift as it allows the lift component in that direction, parasitic drag, and aerodynamic drag to be ignored as well as eliminating the efficacy of the centreboard from the equation. However, all boats will drift to some extent (Figure 16) and in a future simulation, this effect could be explored. Incorporating this would likely result in the optimal points of sail shifting closer to the true wind to compensate for drift.

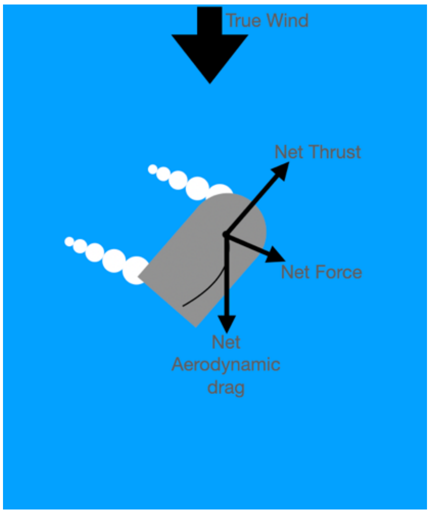

Figure 14

Many boats are designed such that as they increase their speed they can hydroplane. Water is deflected downwards by the bottom pushing the hull up out of the water (Figure 17). This force would have the effect of reducing the drag constant since a lower surface area is immersed in water. As discussed, decreasing drag increases the optimal angle to the wind but since this effect only kicks in at certain velocities, it would be interesting to look at how the optimal point of sail changes as the boat accelerates.

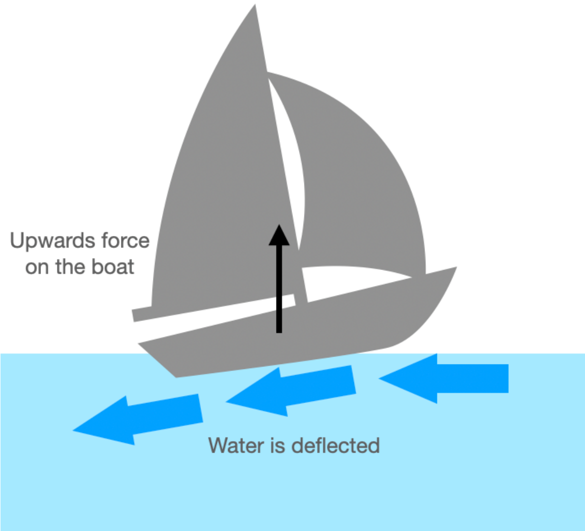

Figure 15

Conclusion

In summary, even a relatively simple mathematical model of a dinghy contains too many variables to determine anything closer than a range of optimal points of sail. To calculate the exact optimal point of sail constant analysis must be performed as the boat is moving with wind data, acceleration and drift all being considered. To do all that intuitively is the skill of sailing.

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Appendix

```python
"""
Created on Wed Oct 27 02:06:08 2021

@author: Henry Hollingworth
"""

from matplotlib import pyplot as plt
from matplotlib import patches as mpatches
from numpy import array
import math
import numpy as np

def absoluteval(a):
    hyp = math.sqrt(a[0]**2 + a[1]**2)
    return hyp

def lift(apparent_wind, boat_bearing, up, lift_coefficient):
    lift = ((absoluteval(apparent_wind)**2)*lift_coefficient)
    vector_lift = lift*(np.cross((-1*apparent_wind), up)/absoluteval(apparent_wind))
    thrust_force = boat_bearing*absoluteval(vector_lift)*vector_lift.dot(boat_bearing)/
(absoluteval(vector_lift)*absoluteval(boat_bearing))
    return thrust_force

def drag(boat_velocity, drag_coeffecient, boat_bearing):
    drag= -1*((absoluteval(boat_velocity)**2)*drag_coeffecient)
    dragv= drag*boat_bearing
    return dragv

def acceleration(FD,FL,boat_mass):
    FR=FL+FD
    a=FR/boat_mass
    return a

def newspeed(acceleration,boat_velocity,time_period):
    newv=boat_velocity+acceleration*time_period
    return newv

lift_coefficient = 0.001
drag_coefficient = 0.1
boat_mass=1
up = array([0,0,1])
true_wind = array([0, -10, 0])
time_period=1

x=[]
y=[]
z=[]

for j in range(90):
    boat_velocity=array([0,0,0])
    FD=0
    FL=0
    time=0
    boat_bearing=array([math.sin(math.radians(j)),math.cos(math.radians(j)),0])

    for i in range(300):
        apparent_wind = true_wind - boat_velocity

        FL=lift(apparent_wind, boat_bearing, up, lift_coefficient)
        FD=drag(boat_velocity, drag_coefficient, boat_bearing)
        boat_velocity=newspeed(acceleration(FD,FL,boat_mass), boat_velocity, time_period)
        time = time+time_period

    x.append(j)
    y.append(boat_velocity[1])
    z.append(absoluteval(boat_velocity))

print(y.index(np.max(y)),np.max(y))
plt.title('The Effect of Point of Sail on Upwind Boatspeed')
plt.xlabel('Angle to the Wind/°')
plt.ylabel('Velocity/knots')
red_patch = mpatches.Patch(color='red', label='Total Velocity')
green_patch = mpatches.Patch(color='green', label='Upwind Velocity')
```

```
plt.legend(handles=[red_patch,green_patch])

plt.plot(x,y, c='g',)
plt.plot(x,z, c='r')
```